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The cumulant approach for investigating the noise influence on mode-locking bifurcations

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Abstract. The influence of noise on mode-locking bifurcations is investigated for the circle map and for coupled logistic maps. The technique based on the cumulant expansion is used for the bifurcation analysis of these systems. It is shown that this cumulant analysis in Gaussian approximation provides a suitable description of the influence of weak noise. We find that the universal scaling properties for the circle map in the critical point for the golden-mean sequence are the same as those obtained analytically using the path integral technique. We find the same scaling behaviour in the case of weak multiplicative noise.

One of the typical routes to chaos is the transition from quasi-periodic oscillations to a chaotic attractor, the so-called Ruelle–Takens route [1, 2]. This can be observed in a wide variety of physical systems, from electrical circuits [3] to the turbulent motion of fluids [4]. During this transition the quasi-periodic motion on the torus is replaced by a chaotic motion whereby the torus is destroyed. Moreover, this route is accompanied by a sequence of mode-locking bifurcations on a torus.

The most popular model for the investigation of the universal properties for this phenomenon is the circle map:

$$\theta_{k+1} = \theta_k + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_k) \qquad \bmod 1. \tag{1}$$

The map (1) may be considered as a nonlinear transformation of the phase of an oscillator of period one. The parameters Ω and K correspond to the ratio of the unperturbed frequencies and the strength of the nonlinearity respectively. In the supercritical case ($K > 1$), map (1) is non-invertible and may exhibit chaotic behaviour.

One of the main methods used to study the properties of quasi-periodic motions is the approximation of the irrational frequency ratio Ω by rationales using a continued fraction representation. Let us consider the mode-locking sequence generated by the Fibonacci numbers

$$F_{n+1} = F_n + F_{n-1} \qquad F_n = 1, 1, 2, 3, 5, 8, \ldots$$

with the rotation numbers ω_n:

$$\omega_n = \frac{F_n}{F_{n+1}} = \frac{1}{1 + F_{n-1}/F_n}. \tag{2}$$

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The limit of this sequence ω_n is the golden mean as n tends to infinity:

$$\omega^* = \lim_{n\to\infty} \omega_n = \omega^* = \frac{\sqrt{5}+1}{2}. \tag{3}$$

In the plane of the parameters (Ω, K) this corresponds to the existence of tongues which bound the mode-locking regions with rotation numbers ω_n. The scaling properties of the width of these tongues $\Delta\Omega_n$ have been investigated in the critical case ($K = 1$) [5]. The width $\Delta\Omega_n$ generated by the sequence (2) displays a self-similar structure:

$$\Delta\Omega_n \propto \delta^{-n} \tag{4}$$

where δ is the universal constant $\delta = 2.8336\ldots$.

The main purpose of this paper is to study the influence of noise on such mode-locking bifurcations. As soon as noise is taken into account we have to consider the statistical quantities of a process and the quantitative changes in them instead of the limit sets in the phase space and their bifurcations in the case of a deterministic system [6, 7]. Such averaged quantities may be the stationary probability density, power spectrum, correlation function, etc. In mathematical language this means that the stochastic equations or the appropriate kinetic equations have to be solved. The bifurcation analysis of such types of equation is a more difficult problem than in the deterministic case.

The problem of noise influence on dynamical systems was first pointed out in the work of Pontryagin *et al* [8]; they used the formalism of Markov processes to study the influence of the external noise on dynamical systems.

Freidlin and Wentzell [9] proposed a theory of weak random perturbations of dynamical systems based on the concept of quasi-potentials which has been extended to chaotic systems [7, 10–15]. This theory provides a correct description of both the stationary probability of the process [7, 10–14] and the noise-induced hopping dynamics [14, 15]. If σ^2 is the intensity of weak external noise ($\sigma^2 \ll 1$), then according to this approach the stationary probability density of a system $p(x, \sigma)$ can be expressed via the quasi-potential $\Phi(x)$ as $p(x, \sigma) \propto \exp(-\Phi(x)/\sigma^2)$. Then a bifurcation can be understood as being a change of the number of extrema of the quasi-potential or the appropriate probability density. This quasi-potential theory provides a rigourous analysis of the scaling properties of dynamical systems and was shown, for example, for period-doubling bifurcations [11, 12] and mode-locking bifurcation [13]. The main advantage of this technique is that it is also applicable to non-Gaussian perturbations. However, the computation of bifurcational lines as well as the continuation of a stationary solution in parameter space is still a non-trivial numerical problem.

In this paper we use another approach based on the cumulant expansion of stochastic systems. The cumulant approach assumes the transition from stochastic equations (or from the appropriate kinetic equation) to deterministic equations which describe the evolution of the cumulants of a stochastic process (cumulant equations) [16]. The main problem that arises is due to the chain of cumulant equations being unclosed because the original stochastic system is nonlinear; that is, the equation for the n-order cumulant also includes higher order cumulants $n+1, n+2, \ldots$. To close this chain of the cumulant equations we use approximations which take into account only a finite number of cumulants [16]. After that closure we can, therefore, carry out an ordinary bifurcation analysis [17, 18] of the cumulant equations. The bifurcations of the steady states of the cumulant equations correspond to the qualitative changes of the shape of the stationary probability density of a Markov process. A rather simple approximation which includes first- and second-order cumulants is the Gaussian approximation; it provides a correct description of the behaviour of a stochastic system with well separated potential minima. Nevertheless, as has recently been shown this approximation often describes the influence of weak noise on the bifurcation analysis of a phase transition induced by coloured noise [19] and on the analysis of period-doubling bifurcations in the presence of noise [20]. As we will show, the cumulant equations in the Gaussian approximation reflect the main features of the stationary probability density even near bifurcation points.

As a study has already been made for the deterministic case, we focus mainly on the circle map using the sequence of Fibonacci numbers of mode-lockings. In the first part we investigate the scaling properties in the critical case ($K = 1$) under the influence of additive noise using the method of cumulant equations. This method also allows us to treat the case of multiplicative noise in the same manner; we will outline this in the second part. Finally, we apply this method to one special resonance tongue for two coupled logistic maps.

In the simplest case the influence of noise on a system can be considered by adding a source of white noise ξ_k:

$$\theta_{k+1} = \theta_k + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_k) + \sigma\xi_k \tag{5}$$

where ξ_k is Gaussian white noise with a zero mean value. If some noise with an intensity σ is added to the map (1) then the tongues become narrower and there exists a maximum value of the noise intensity $\sigma_n^{\max}$ (for each of w_n) beyond which it is impossible to resolve those resonances. The universal scaling law for the noise intensity $\sigma_n^{\max}$ in the critical point $K = 1$ yields [5]

$$\sigma_n^{\max} \propto \beta^{-n} \tag{6}$$

where β is the universal scaling constant $\beta = 2.306\ldots$. This result was obtained analytically by using the functional path integral approach [21] and the quasi-potential method [13].

Now we apply the cumulant approach to this problem. Let us introduce the notions for the cumulants:

$$x_k \equiv \langle\theta_k\rangle \qquad y_k \equiv \langle\theta_k^2\rangle - \langle\theta_k\rangle^2 \tag{7}$$

where the brackets $\langle\cdot\rangle$ denote the averaging over realizations of the stochastic process ξ_k. We derive the equations for the cumulants in the Gaussian approximation directly from the stochastic map (5). Taking into account that in the Gaussian approximation $\langle\sin(2\pi\theta_k)\rangle = \exp(-2\pi^2 y_k)\sin(2\pi x_k)$ we obtain

$$\begin{aligned}
x_{k+1} &= x_k + \Omega - \frac{K}{2\pi}\exp(-2\pi^2 y_k)\sin(2\pi x_k) \\
y_{k+1} &= y_k + \frac{K^2}{4\pi^2}\left(\tfrac{1}{2}(1 - \exp(-8\pi^2 y_k)\cos(4\pi x_k)) - \exp(-4\pi^2 y_k)\sin(2\pi x_k)^2\right) \\
&\quad -2K\exp(-2\pi^2 y_k)\cos(2\pi x_k)y_k + \sigma^2.
\end{aligned} \tag{8}$$

In the weak noise limit we can simplify this map by neglecting the terms of an order higher than y_k. This means that in the second equation only terms linear in y_k are taken into account:

$$\begin{aligned}
x_{k+1} &= x_k + \Omega - \frac{K}{2\pi}(1 - 2\pi^2 y_k)\sin(2\pi x_k) \\
y_{k+1} &= (1 - K\cos(2\pi x_k))^2 y_k + \sigma^2.
\end{aligned} \tag{9}$$

The initial conditions for the cumulant map (9) are $x_0 = \theta_0$ and $y_0 = 0$, where θ_0 is the fixed point of the appropriate deterministic map (1). Note that this dynamical system involves a new parameter—the noise intensity σ. Therefore, we have to consider our system in the extended parameter space (K, Ω, σ).

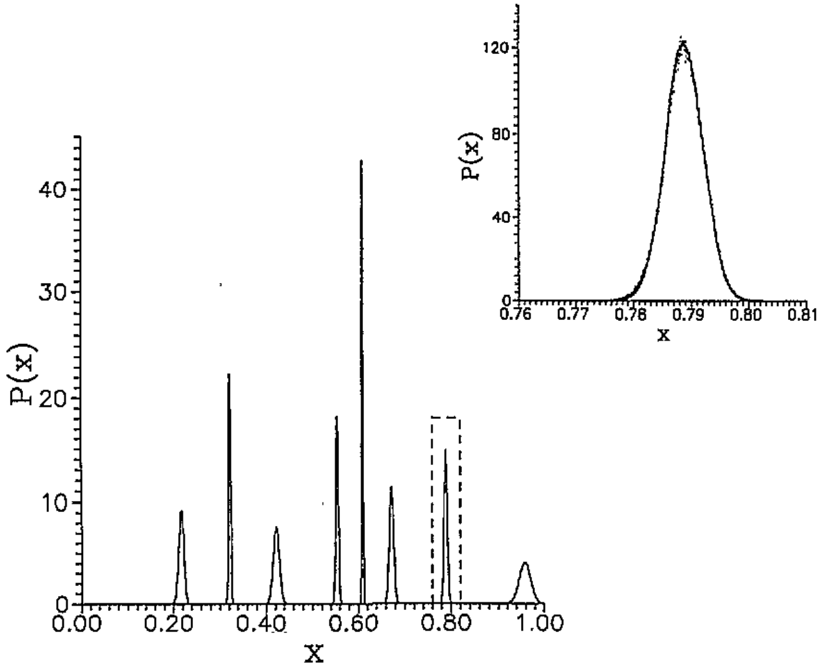

Figure 1. The probability density of the map (5) obtained from numerical simulations. (a) Full structure. (b) Marked maximum; dots correspond to the numerical simulation and the solid curve corresponds to the Gaussian approximation (10).

Firstly, in figure 1(a) we show the result of the numerical calculation of the probability density $P(\theta)$ for the original stochastic map (5). The parameters are ($\Omega = 0.61, K = 1.0, \sigma = 10^{-3}$) which correspond to the mode-locking resonance with the rotation number $\omega_5 = \frac{5}{8}$. Because of the period eight cycle on the torus the probability density consists of 8 maxima and as shown in figure 1(b) each of these maxima can be approximated by the Gaussian distribution $P_{\mathrm{G}}^{(i)}(\theta)$:

$$P_{\mathrm{G}}^{(i)}(\theta) = \frac{1}{\sqrt{2\pi y_0^{(i)}}} \exp\left(-(\theta - x_0^{(i)})^2/(2y_0^{(i)})\right) \tag{10}$$

where $x_0^{(i)}$ and $y_0^{(i)}$ are the coordinates of the ith component of the fixed point of cumulant map (9) for the same parameter values. The same pictures can be obtained for other rotation numbers. Thus, we have indications that the Gaussian approximation gives an appropriate description of the structure of the probability density.

Secondly, we perform the bifurcation analysis of the cumulant map (9) in the parameter plane (Ω, σ) along the sequence of rotation numbers leading to the golden mean. For this purpose we used the software LOCBIF [18]. The bifurcation lines plotted in figure 2 correspond to the birth of resonant cycles with rotation numbers $\omega_n = \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$ and $\frac{8}{13}$. The condition for these bifurcations is that there exists equality of one of the characteristic multipliers of the fixed point to +1. From this diagram it is obvious that boundary values of the noise intensity $\sigma_n^{\max}$ exist beyond where the appropriate mode-locking with rotation number ω_n can be observed. In figure 3 we show the evolution of the numerically obtained probability density for the resonance $\omega_5 = \frac{5}{8}$ ($\Omega = 0.61$) with increasing values of the noise intensity $\sigma^2 = 10^{-5}, 4 \times 10^{-5}, 7 \times 10^{-5}$. These values of the noise intensity correspond

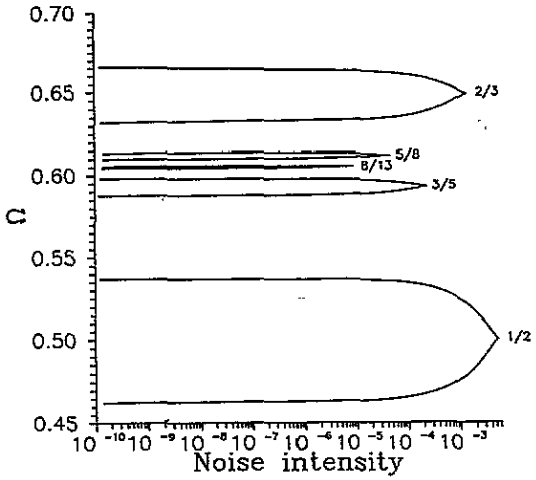

Figure 2. Bifurcation diagram of the cumulant map (9) ($\gamma = 0.292$, $K = 1$).

to the points on the bifurcation diagram (figure 2) inside the region of existence of this resonance, the boundary, and just beyond the boundary. Observe from this figure that noise-induced transitions appear between wells of the appropriate potential. This, of course, cannot be described by Gaussian approximation. Nevertheless, these changes of modality of the probability density are in good agreement with the predictions of the Gaussian cumulant analysis.

It is essential to mention that the sequence $\sigma_n^{\max}$ satisfies the scaling law $\sigma_n^{\max} \propto \beta_{\text{cum}}^{-n}$ with $\beta_{\text{cum}} \approx 2.23$ (cf figure 4). Note that the cumulant approach gives a scaling constant which is in good agreement with the theoretical one.

If we consider the full set of cumulants then we can correctly describe the hopping dynamics but we cannot observe any bifurcations as mentioned in [22]. In the absence of noise in the resonance case we have several locally stable fixed points of the map (1) which are separated by saddle points. As soon as Gaussian noise is added, we have only one globally-stable stationary solution (the stationary probability density) of the appropriate Frobenius–Perron equation. If we vary the parameters of the system we only obtain changes in the shape of the stationary probability density (which can be understood as bifurcation transitions [6]) but no bifurcations in a rigourous mathematical sense. These bifurcations can be determined if we approximate each maximum of the probability density by a model distribution, in the simplest case by the Gaussian distribution. The location of the maxima of the probability density will then be defined mainly by the first cumulant of the cumulant map (9). The second cumulant allows us to take into account the influence of noise in the first order by the parameter σ^2. The bifurcation picture remains qualitatively still the same if we include high-order cumulants (we include two more cumulants of third and fourth orders).

In the same manner we can consider the case of coloured noise, i.e. when the correlation function of ξ_k is not a δ function of the time difference. In this case the process θ_k is a non-Markov process. The usual approach is based on the extension of the stochastic map (5) by introducing an additional stochastic map which describes the coloured noise. For example we can use a discrete analogue of the Ornstein–Uhlenbeck process, an autoregressive process of first order, as has been done for the period-doubling bifurcation [20].

This approach to the study of cumulant equations also allows us to treat the case

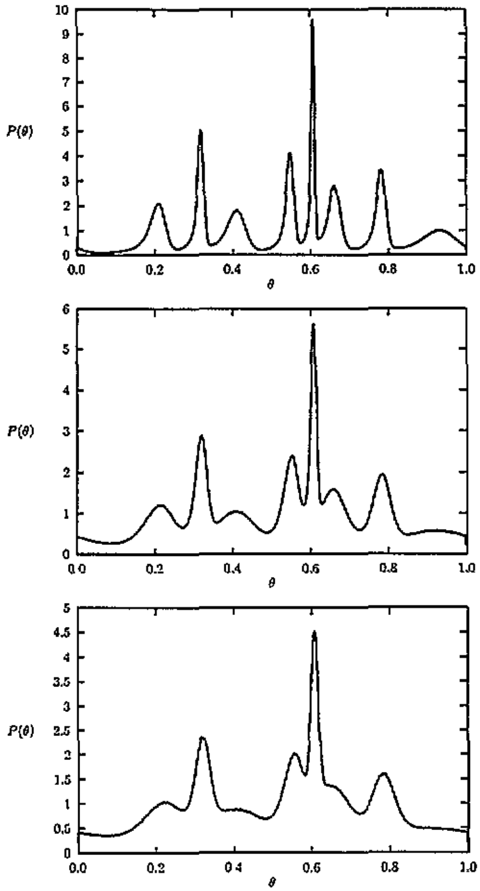

Figure 3. The probability densities of the map (5) obtained from numerical simulations for $\Omega = 0.61$ and for the noise intensity $\sigma^2 = 10^{-5}, 4 \times 10^{-5}, 7 \times 10^{-5}$ (from above).

of multiplicative noise. To our knowledge there are no analytical results concerning the influence of multiplicative noise on mode-locking bifurcations. Let us now consider this case which can be achieved by a stochastic modulation of the parameter K of the map (1). In this case we can write the stochastic map in the form:

$$\theta_{k+1} = \theta_k + \Omega - \frac{K + \xi_k}{2\pi} \sin(2\pi\theta_k). \tag{11}$$

In the limit of weak noise the approach outlined above yields the cumulant equations

$$x_{k+1} = x_k + \Omega - \frac{K}{2\pi}(1 - 2\pi^2 y_k) \sin(2\pi x_k)$$
$$y_{k+1} = (1 - K\cos(2\pi x_k))^2 y_k + \frac{K^2}{4\pi^2}\sigma^2 \sin(2\pi x_k). \tag{12}$$

The result of the bifurcation analysis of this system for the Fibonacci sequence of mode-lockings is similar to the result in the case of additive noise (see figure 2). We obtain the same type of bifurcation diagram with a scaling constant $\beta_{\text{cum}} \approx 2.26$. Thus, the universal scaling behaviour (6) has been observed in the case of multiplicative noise as well as for additive noise with approximately the same scaling constant. We expect a difference

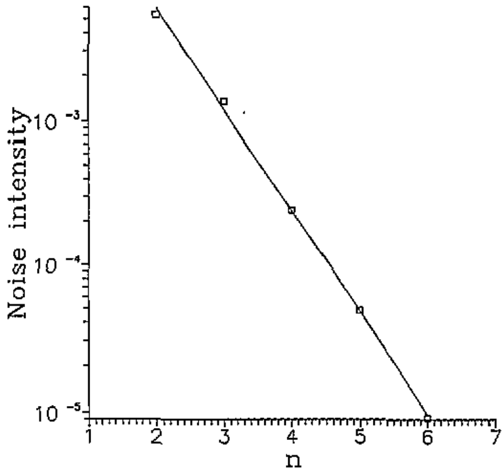

Figure 4. σ_n^{max} plotted against n (squares) and its fit by the law β_{cum}^{-n}, $\beta_{cum} = 2.23$.

between the influence of additive noise and multiplicative noise in the case of noise with moderate intensity [22].

We now consider another model, namely two coupled logistic maps [3] with additive noise sources:

$$x_{n+1} = 1 - ax_n^2 + \gamma(y - x) + \sigma\xi_n^{(1)} \qquad y_{n+1} = 1 - ay_n^2 + \gamma(x - y) + \sigma\xi_n^{(2)}. \tag{13}$$

The noise sources $\xi_n^{(1,2)}$ are statistically independent, white random processes with a zero mean value symmetrically distributed in the region $[-\epsilon, \epsilon], \epsilon \ll 1$:

$$\langle \xi_n^{(i)} \xi_{n+m}^{(j)} \rangle = \delta_{ij}\delta(m). \tag{14}$$

Again, let us introduce the notions for the cumulants of the first and second order:

$$\begin{aligned} &X_n \equiv \langle x_n \rangle \qquad Y_n \equiv \langle y_n \rangle \\ &U_n \equiv \langle x_n^2 \rangle - \langle x_n \rangle^2 \qquad V_n \equiv \langle y_n^2 \rangle - \langle y_n \rangle^2 \\ &Z_n \equiv \langle x_n y_n \rangle - \langle x_n \rangle \langle y_n \rangle. \end{aligned}$$

In the Gaussian approximation we obtain the following five-dimensional map for the cumulants:

$$\begin{aligned} X_{n+1} &= 1 - a(X_n^2 + U_n) + \gamma(Y_n - X_n) \\ Y_{n+1} &= 1 - a(Y_n^2 + V_n) + \gamma(X_n - Y_n) \\ U_{n+1} &= (2aX_n)^2 U_n + \gamma^2(U_n + V_n - 2Z_n) - 4a\gamma X_n(Z_n - U_n) + \sigma^2 \\ V_{n+1} &= (2aY_n)^2 V_n + \gamma^2(U_n + V_n - 2Z_n) - 4a\gamma Y_n(Z_n - V_n) + \sigma^2 \\ Z_{n+1} &= (4a^2 X_n Y_n + 2a\gamma(X_n + Y_n) + 2\gamma^2)Z_n - (2a\gamma X_n + \gamma^2)U_n - (2a\gamma Y_n + \gamma^2)V_n. \end{aligned} \tag{15}$$

In the absence of noise the bifurcations in this model are well known (cf [3]). In the (γ, a) plane there is a line of bifurcations to a torus. On this line there is a countable set of bifurcation points of codimension two which correspond to the mode-locking resonances with rational rotation numbers. Since the rotation number on the line of torus bifurcations is a function of two variables a and γ, it is more difficult to obtain the scaling behaviour as in the case of the circle map. That is why we only consider in detail the resonance with the rotation number $\omega = \frac{2}{5}$. In figure 5 (curve 1) we show a part of the bifurcation diagram in

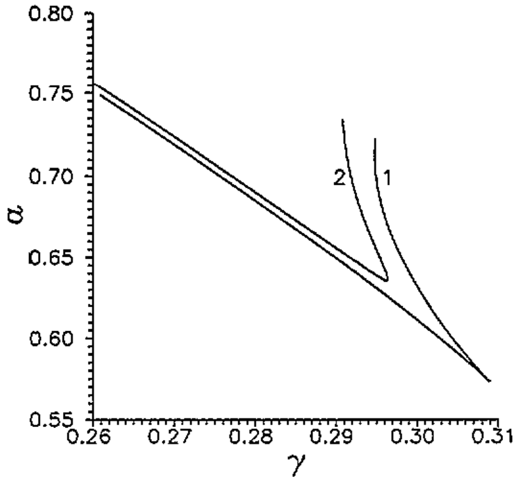

Figure 5. Bifurcation diagram of the map (15) on the parameter plane (γ, a) with $\sigma = 0$ (curve 1) and $\sigma = 10^{-3}$ (curve 2).

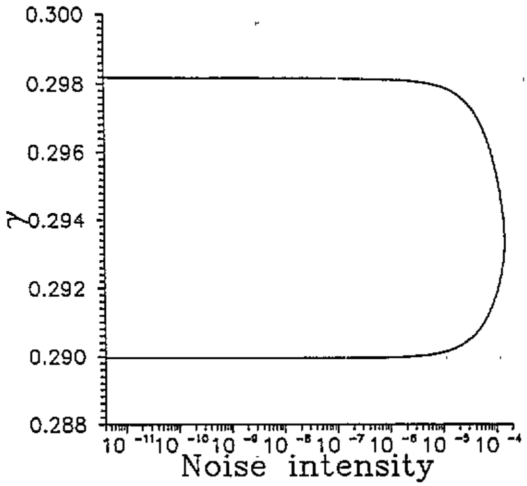

Figure 6. Bifurcation diagram of the map (15) on the plane (σ^2, γ), $(a = 0.65)$.

the parameter plane (γ, a) with the resonance tongue $\omega = \frac{2}{5}$. This tongue is formed by the bifurcation lines which correspond to a saddle-node bifurcation of a cycle of period ten.

If we impose weak noise on the system then the mode-locking resonance p/q will correspond to the cycle of the cumulant map (15) of the period pq. This resonance tongue on the parameter plane (γ, a) becomes narrower if noise is taken into account (see figure 5, curve 2). Let us consider the bifurcation diagram on the parameter plane (σ, γ) (see figure 6). Observe from this figure that again there exists a value of noise intensity σ_m which bounds the region of existence of this resonance.

In conclusion, we have shown that the technique of cumulant expansion can be applied to the analysis of mode-locking bifurcations even in Gaussian approximation. This method provides both a qualitative description of the bifurcation behaviour and a suitable agreement with theoretical results on the universal scaling properties. The advantage of this approach lies in the consideration of a deterministic system instead of the original stochastic system. As a consequence, the ordinary bifurcation analysis can be carried out in the extended parameter space of the system. An important advantage of this approach is that we can treat high-dimensional maps as well as the case of multiplicative noise or coloured noise [20].

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